Section 6

Modelling Demography and the Regional Economy: An Interregional General Equilibrium Modelling Framework

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1. Abstract[1]

An ageing population is a challenge facing most economies in OECD countries today. Low fertility and longer average living ages will hollow out the economic basis for the majority of the OECD countries. In this paper a regional model including the age impact on the regional and local/urban economy is presented.

The age problem involves both an income/supply side and an expenditure/demand side. Level of income in a region/local area is determined by the productive part of the population and their productivity. As the share of population of older people increases, which has a below average productivity and labour participation rate, the process of ageing for most regions will lead to lower production and lower income in regions in most of the OECD countries.

On the demand side the consumption by old people (either the private consumption or the governmental consumption) will change. For the older population the demand will not increase, because increasing individual governmental consumption for age-specific commodities such as medicine and hospitals will be outweighed by a decrease in private consumption. But for the very old population, the total consumption will increase because the increase in cost in keeping the very old alive is often higher than the decrease in private consumption.

In this paper a comparative static, general, interregional, equilibrium model for local or urban economies is presented. The aim of the model is to illustrate the impact of comparative static shifts in the age structure of the population in terms of impact on average real disposable income (in total and by age group). The model includes the division of the population, the labour force, the employment and unemployment rate into age groups. The model includes the local, the interregional and the international levels, reflecting different equilibrating processes, where excess supply and demand can exist in certain markets, whereas equilibrium does not prevail in other markets. The theoretical model is formulated in changes and is solved for a quantity and a price side. Multipliers for the quantity and price side of the local economy are derived. The signs and the magnitudes for different types of areas of the ageing process are discussed.

The model can be formulated in different ways reflecting different time horizons and different regimes for regulation and barriers for interregional and international interaction: In the long run, interregional migration can equilibrate excess supply and demand in the domestic labour markets, whereas equilibration in the international demand and supply of labour depends on the regulation of international migration flows. Different specification of the model

1 I want to thank senior researcher Frank Jensen for comments and advice in the work with the mathematical formulation of the general interregional model in difference.

reflecting different regimes of interregional and international interaction and the time horizon in the equilibrating process are discussed.

The paper is organised as follows. In section 2 the basic model is presented, while section 3 includes the basic equations. The model is solved in section 4 and an interpretation of the model can be found in section 5 including a discussion of an alternative specification of the model reflecting different time horizons and different regimes for interregional and international interaction. Section 6 concludes the paper.

2. The model

In Madsen (2007a & 2007b) the two-by-two-by-two principle is used as the basic structure of the general, interregional, static quantity and price models in order to give a general overall picture of a local economy. In this paper, the model is transformed into a model in changes. In the model the labour market is modelled as an age-divided market. Non-linear functions in labour demand, productivity and labour-participation rates are included. Now, the two-by-two-by-two principle is presented. After that a model, which (compared to the general interregional linear quantity and price models) has been slightly simplified including non-linear labour demand, productivity and labour-participation rate functions, is examined. The model, which is formulated in changes, is solved analytically. Finally, the solutions to the modified quantity and the price models are presented and multipliers are examined and used to classify local areas into different archetypes.

2.1. The two-by-two-by-two principle – Basic concepts and dimensions in the interregional model

There are three fundamental dimensions in the general, interregional, quantity and price model, following the two-by-two-by-two principle (Madsen & Jensen-Butler 2005, Madsen 2007a & 2007b). First, both producers and households are represented in the model. Second, two markets (a commodity market and a factor market) are included in the general model. Third, interaction between markets and actors includes information on origins and destinations. For both actors and markets basic geographical concepts have been used as well as social accounting concepts for activities.

In factor markets supply and demand of production factors are to be found. Demand for production factors (g) is determined by production by sector (j) at the place of production (P). Factor demand by sector is transformed into factor demand by type of production factor (g). On the supply side, supply of production factors by type of institution (h) is transformed into supply by type of production factor (g). Supply of a production factor is related to the place of residence of the institution (R). The factor market is geographically assigned to the market place for factors (Q) (see figure 1).

Completing the presentation of the general model based on the two-by-two-by-two principle, in the commodity market there is a distinction between place of residence (R), the market place for commodities (S) and place of production (P). The market place for commodities links the demand for the commodity (from place of residence to the market place for commodities) to the supply of the commodity (from place of production to the market place for commodities). Before the transformation to the market place for commodities, the demand for commodities is transformed from institutional group (h) to commodity (i). On the supply side, production by sector (j) is transformed into production by commodity (i) and then supply is related geographically to the market place for commodities (S).

In comparison with this general, interregional, quantity and price model based upon the two-by-two-by-two principle, a 3-dimensional model has been used in this paper. The place of factor market (Q) and type of institution (h) have been omitted.

Second, this model is an open model in trade only opposite the general interregional model, which includes both domestic and foreign sectors in all markets. Thereby, it omits other types of international interaction, such as cross border commuting (income flows to and from abroad through commuting), border shopping, which includes one-day tourist expenditure in both directions, and tourism are excluded.

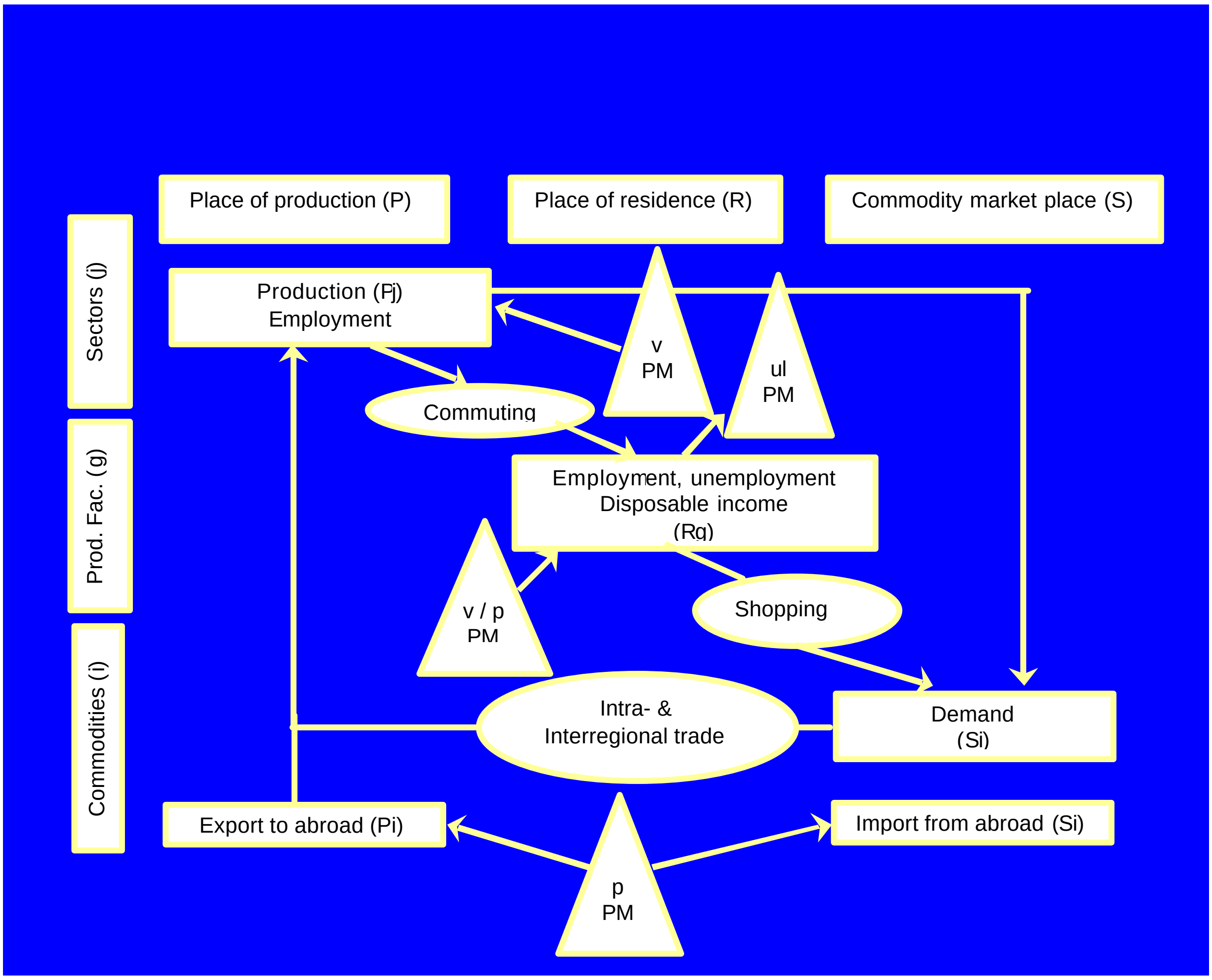
These exclusions are justified, because the model becomes smaller and more operational without loss of generality. Simplifications are necessary as one of the main aims of the model is to introduce more complex and non-linear behaviour equations for factor demand and productivity and labour-participation rate equations.

2.2. The general interregional quantity and the price submodels – a graphical presentation

The model by Madsen (2007a & 2007b) includes two submodels, the quantity and the price model. The two submodels are sequential models or models with a circular causal structure. In the quantity circuit the well known effects from demand on supply (income) and from income to demand are included. In the price circuit the spillover and feedback effects of cost and price changes using an adding-up principle based upon the assumption of perfect competition are modelled.

Taking the 3-dimensional model as point of departure, the two submodels are sequential models, which are represented in figure 1 and 2 below.

Figure 1. The real circle – the quantity submodel



Note: PM: Price submodel; ul: unemployment; p: prices; v: income rates.

The horizontal dimension is spatial and includes place of production (P), place of residence (R) and place of commodity market (S). Production activity is related to place of production. Factor rewards and income to production factors are related to place of residence and demand for commodities is assigned to place of commodity market. The vertical dimension follows with its threefold division the general structure of a SAM model. Production is related to activities (j); factor incomes are related to factors of production with labour classified by age (g), commodities are related to the supply and demand for commodities (i).

The real circuit – or the quantity model – corresponds to a demand-driven Keynesian model and moves sequentially and clockwise in figure 1. Starting in the upper left corner (Pj), production generates intermediate consumption demand and employment by sectors (j) at the place of production (P). The employment is transformed from sectors (j) to age groups (g) determined by relative income and from place of production (P) to place of residence (R) through a commuting model. Labour force at the place of residence depends upon population by age (g) and activity rates by age, which in turn depends upon age specific income rates. Labour force and employment determine the unemployment by age (g) at place of residence (R).

Real disposable income by age groups at the place of residence is determined by private consumption prices, employment and income rates (Rg). Real disposable income and commodity-specific consumption rates are the basis for determination of private consumption by place of residence (R) and by commodity (i). Private consumption is assigned to place of commodity market (Si) using a shopping model. Governmental consumption is determined by population by age (g) and the age and commodity (i) specific consumption rates at the place of residence (R). Governmental consumption by commodity (i) is transformed into commodity market place (Si), where the consumption takes place. Intermediate consumption at place of production (P) and by sector (j) is determined by gross output and transformed to place of commodity market (Si). Intermediate, private and governmental consumption together with investment constitute the total local demand for commodities (Si), by commodity. Local demand is met by imports from other regions and from abroad in addition to local production (Si). Through a trade model exports abroad and to other regions and production for the region itself are determined (Pi). Gross output by commodity is determined by this demand. Through a reverse make matrix the cycle returns to production by sector (Pj).

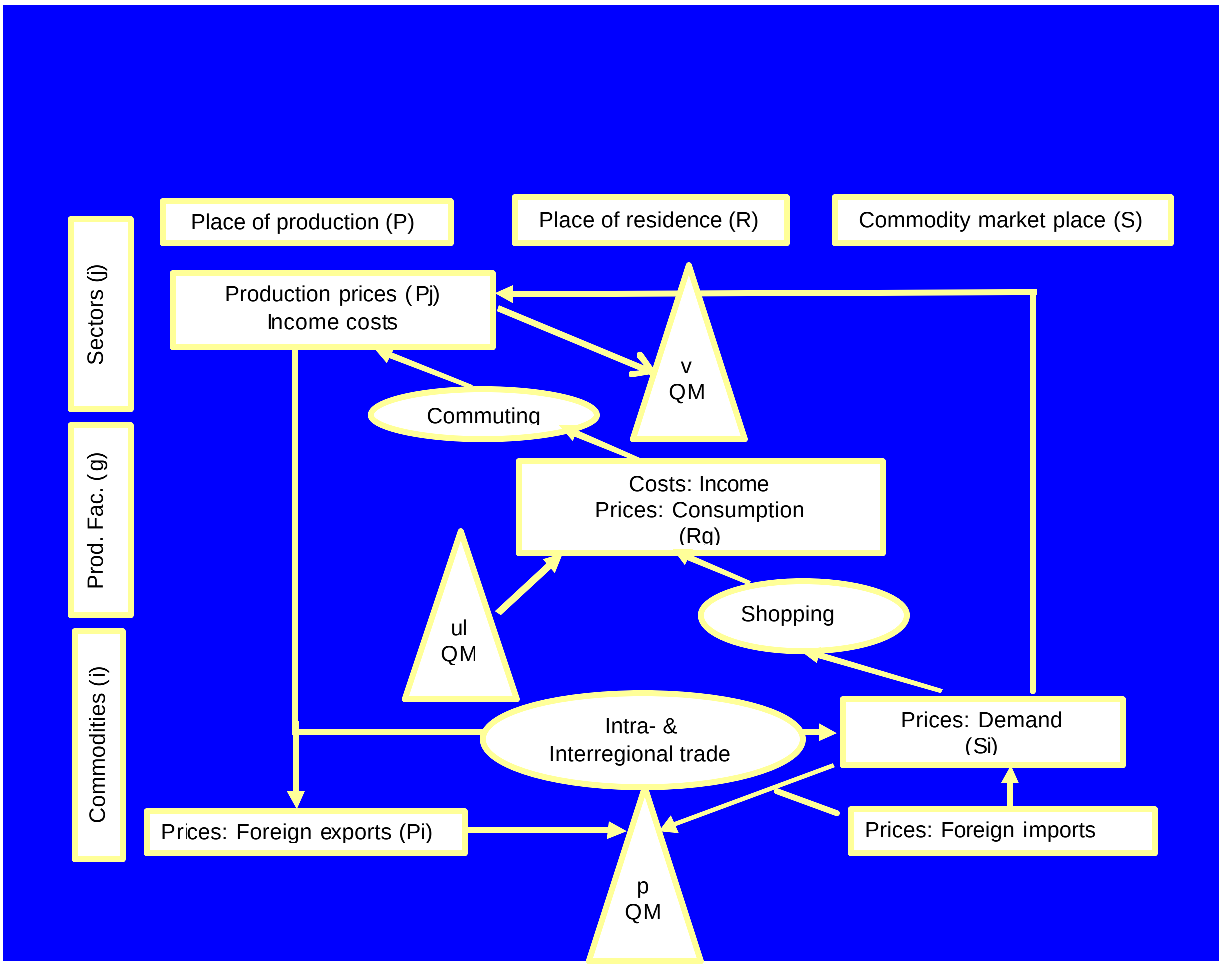
Economic activity in the real circle or the quantity model is affected by changes in prices and income: The anticlockwise cost/price circuit (or the price model) shown in Figure 2 corresponds to this dual problem. In the upper left corner a price index for production by place of production (P) and by sector (j) is determined by a cost approach (intermediate consumption and value added, Pj). Income rates affect the composition of labour in the production (from j to g), which in turn affect the productivity by sector (from g to j). Together with prices (price indices) on intermediate consumption income costs (income rates) affect prices (price indices) of the local production (Pj). Through a make matrix, sector prices by sector are transformed into sector prices by commodity (i), which through relative changes in local competitiveness affects exports abroad (Pi). These are then transformed from place of production to place of demand (Si). Commodities for intermediate consumption enter into the next step in the production chain, through transformations from place of commodity market (S) to place of production (P) and from commodity (i) to sectors (j), determining prices of production (Pj). These prices are spread further in a round-by-round distribution process following a sequential modelling routine.

Finally, prices on private consumption are transformed from place of commodity market (S) to place of residence (R) and from commodities (i) to age groups (g). At the labour market

(Rg), nominal income rates by age groups at the place of residence are determined as functions of unemployment and private consumption prices. Unemployment was determined in the quantity model. Income rates are then transformed to place of production (P) according to commuting patterns and according to the age composition of employment by sectors (j). Income is now determined implicitly as income rates multiplied by the employment, which is determined in the quantity model.

In the quantity model, income rates enter into functions for determination of employment by age group by sector (from j to g). This implicitly determines the productivity being a function of employment by age group by sector (Pj).

Figure 2. The cost-price circle – the price submodel



Note: QM: Quantity submodel; ul: unemployment; p: prices; v: income rates.

2.3. Age in the model

As indicated, age enters into the model as a factor variable. Age enters into both the factor market and the commodity market. In the factor market labour supply and demand are divided by age groups. The labour-demand function includes a substitution between different age groups in the production as well as a function for productivity by age groups which becomes a function of the composition of employment by age. On labour-supply side population by age is exogenous, whereas the labour participation rate by age depends upon the age-specific income rates.

In the commodity market the demand side is divided into private consumption and governmental consumption. Consumption rates depend upon age structure, and private consumption is small and governmental consumption is high for old age groups. Private consumption then depends on income and the age-specific consumption rates. Governmental consumption depends on population by age group and the age-specific consumption rates. From production side, age structure influences the competitiveness, because productivity and cost of production depend upon age. This influences demand (export abroad and import from abroad). Ageing involves loss of competitiveness, which in turn leads to lower demand and lower production.

3. The equations for the general interregional quantity and price model for local and urban economies in structural form

In this section, the equations of the model are examined in detail. First, the mathematical notation is presented. Then the quantity and price submodel are presented – based upon the sequential structure of the submodels following the circle in figure 1 and 2. Non-linearities in the quantity model are examined in detail.

3.1. The model – notation

The notation includes such information as variable names, subscripts, superscripts and mathematical operators. In general, the equations in the model involve tensor algebra, which is multi-dimensional matrix algebra. However, most of the notation from 2-dimensional matrix algebra can be used in tensor algebra without further explanation.

The upgrading from matrix to tensor algebra is necessary, because most variables involve one or two regional specifications. For example commuting, which is employment at the place of production and the place of residence by age group, is 3-dimensional. If also age and education and the time axis are included, the tensors will be 6-dimensional.

Variables in the model

The variables in the general interregional model are denoted in the following way:

Variables

b: Use share vector of demand
B: Use coefficient matrix of demand
D: Make coefficient matrix
G: Factor demand and composition coefficient matrices
h: Income vector
H: Income transformation coefficient matrices
J: Commuting coefficient matrix
pu: Price index vector for demand
pv: Income cost index vector
q: Employment vector
S: Shopping matrix
T: Trade coefficient matrix
v: Income rate vector
x: Gross output vector
z: Trade vector

Superscripts

P: Place of production (regional axis)
R: Place of residence (regional axis)

S: Place of commodity market (regional axis)

Subscripts
SAM-axes
j: Sector (SAM-axis)
g: Groups of factors (SAM-axis)
i: Commodity (SAM-axis)
IC: Intermediate consumption
CP: Private consumption
CO: Governmental consumption
I: Investments

Mathematical notation
‘: Transposed
$\circ$: Element to element multiplication
M: Capital letters for matrices (tensors)
v: Lower case letters for vectors
dM/dv: Difference for the matrix M or the vector v

In the equations 1-39 in the following two sections the general local/urban quantity model is presented, where 1-24 are the quantity submodel, whereas equations 25-39 are the price submodel.

3.2. The quantity submodel in structural form
The equations in the quantity submodel go clockwise (see figure 1) and follow the sequential structure described in section 2.2:

$$u^P_{IC,j} = b_{IC} \circ x^P_j \text{..(1)}$$

$$u^P_{IC,i} = B_{IC} u^P_{IC,j} \text{..(2)} \quad \textit{from } j \textit{ to } i$$

$$u^S_{IC,i} = S_{IC} u^P_{IC,i} \text{..(3)} \quad \textit{from } P \textit{ to } S$$

$$g^P_j = l\,(q^P_{j,g})^d \text{..(4)}$$

$$q^P_j = x \circ g^P_j \text{..(5)} \quad \textit{from } j \textit{ to } g$$

$$G = a\,(pv^P_{j,g})^b \text{..(6)}$$

$$q^P_{j,g} = G \circ q^P_j \text{..(7)} \quad \textit{from } j \textit{ to } g$$

$$q^P_g = i' q^P_{j,g} \text{..(8)} \quad \textit{from } P \textit{ to } R$$

$$q^R_g = J q^P_g \text{..(9)} \quad \textit{from } P \textit{ to } R$$

$$lq^R_g = e\,(pv^R_g)^n \text{..(10)}$$

$$l^R_g = lq^R_g \circ pop^R_g \text{..(11)}$$

$$ul^R_g = l^R_g - q^R_g \text{..(12)}$$

$$h^R_g = q^R_g \circ pv^R_g \circ v^R_g \text{..(13)}$$

$$u^R_{CP,g} = b_{CP} \circ h^R_g \circ (pu_{CP,g})^{-1} \text{..................................(14)}$$

$$u^R_{CP,i} = B_{CP} u_{CP,g} \text{..(15)} \quad \textit{from } g \textit{ to } i$$

$$u^S_{CP,i} = S_{CP} u^R_{CP,i} \text{..(16)} \quad \textit{from } R \textit{ to } S$$

$$u^R_{CO,g} = b_{CO} \circ pop^R_g \text{..(17)}$$

$$u^R_{CO,i} = B_{CO} u^R_{CO,g} \text{..(18)} \quad \textit{from } g \textit{ to } i$$

$$u^S_{CO,i} = S_{CO} u^R_{CO,i} \text{..(19)} \quad \textit{from } R \textit{ to } S$$

$$u^S_i = u^S_{IC,i} + u^S_{CP,i} + u^S_{CO,i} + u^S_{I,i} \text{..............................(20)}$$

$$z^{S,D}_i = (i - d^S_i) \circ u^S_i \text{..(21)}$$

$$z^{P,D}_i = T z^{S,D}_i \text{..(22)} \quad \textit{from } S \textit{ to } P$$

$$x^P_i = z^{P,D}_i + z^{P,F}_i \text{..(23)}$$

$$x^P_j = D x^P_i \text{..(24)} \quad \textit{from } i \textit{ to } j$$

Starting in the upper left hand corner at place of production by sector (cell Pj in figure 1) in equation 1 intermediate consumption u^P_{IC} is determined. u is demand and is a vector. The subscript IC indicates intermediate consumption. The superscript shows, that intermediate consumption is determined at the place of production P. Intermediate consumption is a function of the gross output vector x^P_j (by place of production P in fixed prices and by sector j) and the intermediate consumption share vector b_{IC}, which is intermediate consumption as share of production. In equation 2, intermediate consumption by commodity $u^P_{IC,i}$ is determined on the basis of a use matrix for intermediate consumption B_{IC} and the vector for intermediate consumption by sector from equation 1 (u^P_{IC}). In equation 3, intermediate consumption is transformed from place of production $u^P_{IC,i}$ to place of commodity market $u^S_{IC,i}$ using a shopping matrix for intermediate consumption goods (S_{IC}).

Continuing in the upper left corner (cell Pj), production generates employment q^P_j by sector j from gross output by sector and labour content g_j (equation 5). Employment content – or the inverse productivity – by sector is assumed to be a non-linear function of employment by age groups ($q^P_{j,g}$) (equation 4). A change in the composition of employment – from younger to older – will result in a change in labour content or in productivity – in this case an increase in labour content or a reduction in productivity.

In equation 7 employment by age group and by sector is determined by employment by sector vector (q^P_j) and the age by sector composition matrix (G). The share of employment by age group is assumed to be a non-linear function of the relative income rates for different employment age groups. In equation 6 the age composition matrix is function of the relative income rates by employment age group ($pv^P_{j,g}$). In equation 8 employment by age is found by summation.

Then employment by age is transformed from place of production P to place of residence R through a commuting model (from cell Pg to cell Rg, equation 9).

Labour force by age is determined by population and the participation rate (equation 11). The participation rate by age group is determined as a non-linear function of the income rate (pv^R_g) (equation 10).

Unemployment is labour force minus employment at the place of residence (equation 12). Unemployment enters into the determination of the income rate in the price model.

Income and prices on private consumption by age group are the basis for determination of private consumption place of residence (cell Rg) (equation 14). Income by age (equation 13) is found using employment (q^R_g), average income rate (v^R_g) and an income index (pv^R_g). The income cost index (pv^R_g) is determined in the price model.

In equation 14 private consumption u^R_{CP} is determined on the basis of a consumption share vector (b_{CP}) and on prices on private consumption goods (pu^R_{CP}), which in turn are determined in the price submodel.

Private consumption by commodity $u^P_{IC,i}$ is determined on the basis of a use matrix for private consumption (B_{CP}) (equation 15). Private consumption including both tourism (domestic and international) and local private consumption (cell Ri) is assigned to the place of the commodity market (cell Si) using a shopping model for local private consumption and tourism (equation 16).

Governmental consumption (cell Ri) is determined by population by age (u^R_g) and age specific consumption rates (b^R_{CO}, see equation 17). Governmental consumption by commodity $u^R_{CO,i}$ is determined on the basis of a use matrix for governmental consumption (B_{CO}) (equation 18). Through a model for shopping in local government commodities demand is transformed from place of residence to place of commodity market (cell Si) (equation 19).

Intermediate consumption together with private consumption, governmental consumption and investment constitute the total local demand for commodities (cell Si) (equation 20). Local demand is met by imports from abroad and other regions in addition to local production (equation 21). Through a trade model exports to other regions and production for the region itself are determined (equation 22). Adding export abroad, gross output by commodity is determined (cell Pi) (equation 23). Through a reverse Make matrix the cycle returns to production by sector (cell Pj) (equation 24).

3.3. The price submodel in structural form

The price model follows the sequential model described in section 2.2 and goes anticlockwise – see figure 2.

$$px_i^P{}' = px_j^P{}' D \quad \text{.......(25)} \quad \textit{from j to i}$$

$$pz_i^{P,F}{}' = px_i^P{}' \quad and \quad pz_i^{P,D}{}' = px_i^P{}' \quad \text{.......(26)}$$

$$pz_i^{S,D}{}' = pz_i^{P,D}{}' T \quad \text{.......(27)} \quad \textit{from P to S}$$

$$pu_i^S{}' = pz_i^{S,D}{}' \circ (i - d_i^{S,F}{}') + pz_i^{S,F}{}' \circ d_i^{S,F}{}' \quad \text{.......(28)}$$

$$pu_{IC,i}^P{}' = pu_i^S{}' S_{IC} \quad \text{.......(29)} \quad \textit{from S to P}$$

$$pu_{IC,j}^P{}' = pu_{IC,i}^P{}' B_{IC} \quad \text{.......(30)} \quad \textit{from i to j}$$

$$pu_{CP,i}^R{}' = pu_i^S{}' S_{CP} \quad \text{.......(31)} \quad \textit{from S to R}$$

$$pu_{CP,g}^R{}' = pu_{CP,i}^R{}' B_{CP} \quad \text{.......(32)} \quad \textit{from i to g}$$

$$dpvx_g^R{}' = (ul_g^n{}' - ul_g{}')\, c_g^R{}' \quad \text{.......(33)}$$

$$pv_g^R{}' = p_{CP,g}^R{}' \circ b_{CP}{}' + pvx_g^R{}' \circ (i' - b_{CP}{}')' \quad \text{.......(34)}$$

$$pv_g^P{}' = pv_g^R{}' J_g^{P,R} \quad \text{.......(35)} \quad \textit{From Rg to Pg}$$

$$pve_{j,g}^P{}' = pv_g^P{}' \circ G_{j,g}^P \quad \text{.......(36)} \quad \textit{From Pg to Pj}$$

$$pve_j^P{}' = i' pve_{g,j}^P \quad \text{.......(37)}$$

$$pv_j^P{}' = pve_j^P{}' \circ (i' - jx_j^P{}') + pvx_j^P{}' \circ jx_j^P{}' \quad \text{.......(38)}$$

$$px_j^P{}' = pu_{IC,j}^P{}' \circ b_{IC}{}' + pv_j^P{}' \circ (i' - b_{IC}{}') \quad \text{.......(39)}$$

Starting in the upper-left corner prices in production by sector and at place of production are transformed into a price index for production by commodities (equation 25). Here the make matrix D from the quantity model is used as weights. The price index for production by commodities is – in this simplified model – also equal to the price index for exports abroad and the price index for the domestic market (equation 26). The price index for domestic production by place of production is then transformed into a price index for demand for domestically-produced commodities at the place for commodity markets (equation 27). The trade coefficient matrix from the quantity model is used as a weight matrix. To get the price index for total demand by place of commodity market, the foreign import price index is added using the foreign import share by commodity as weights (equation 28). Again, in this simplified model the price index is the same for different types of demand. Intermediate consumption price index by place of production and by commodity is determined using the shopping matrix from the quantity model (equation 29). This in turn is transformed from price indices by commodity to price indices for intermediate consumption by sector using the use matrix for intermediate consumption from the quantity model (equation 30).

The price index for private consumption demand is transformed from place of commodity market to place of residence using the shopping matrix as weights (equation 31). Furthermore, the private consumption price index is transformed into a price index by age groups using the use matrix by age groups as weights (equation 32). This price index for private consumption together with a cost index for income not used for consumption determines the total income cost index (equation 34). The unemployment (from the quantity model) enters into the equation for determination of the cost index for the income, which is not consumed (equation 33). Here the change in income cost index is assumed to depend upon the difference between unemployment and the natural rate of unemployment by age group. In equation 35, the cost index for income by place of production is derived using the commuting coefficients matrix from the quantity model. Further, a cost index for income by age group and sector is derived in equations 36 using the labour content matrix from the quantity model. A cost index for total income at place of production is found in equation 38 by adding the cost index for the endogenously determined income with a cost index for the exogenously determined factor income.

In equation 39 the production price index is determined using a simple adding-up principle on intermediate consumption price index and income index using the cost shares as weights.

3.4. Linking the quantity and the price model – Non-linearities in the quantity model

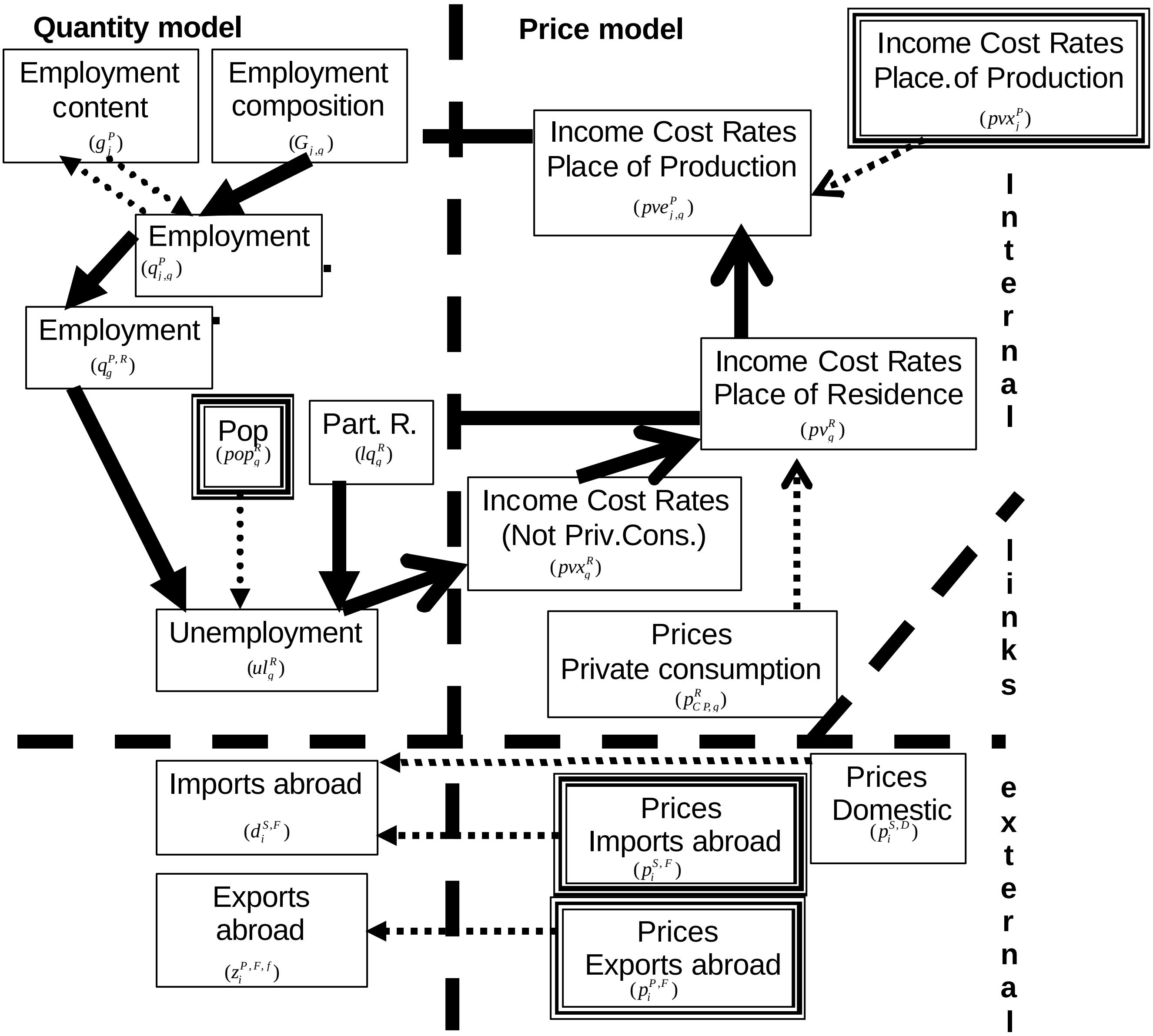
The submodels are linked together such that prices influence quantities and quantities influence prices. The links between the two models can be illustrated in figure 3.

The links are divided into internal links and external links. Internal links are defined as links, which form a simultaneous system covering both models, whereas external links are links where the links are non-simultaneous, where an endogenous variable, which is an exogenous explaining variable in the other submodel, does not enter as explanatory variable in the submodel itself.

The upper part of the diagram shows – in bold – two internal links in the model: First, in the quantity model (equation 10) – shown as the inner bold circle – the labour participation rates by age group and by place of residence depend upon the income rates, which are determined in the price model (equation 34). Labour force is a function of the population and of the labour-participation rates, which in turn together with the employment determine the unemployment (the dashed line in the diagram – see equation 12). Unemployment (the quantity model) together with private consumption prices (the price model) explain the income rates in the price model (equation 30), forming a simultaneous block.

Second, in the quantity model – shown as the outer bold circle in figure 3 – income rates (price model) determine the age composition of employment (equation 5). The age composition explains the labour content (inverse productivity – see equation 6), which – after a number of transformations (equations 7 to 9) – enter into the determination of employment and unemployment by place of residence (the dashed line in the diagram). Unemployment together with private consumption prices enter into the income rate equation in the price model (equation 34). Income rates by place of residence are in the price model transformed from the place of production using employment and commuting variables from the quantity model (the dashed line in the diagram – see equations 35 to 38) forming another simultaneous block covering both submodels.

Figure 3. Links between the quantity and the price models



External links have not been included in the presentation of the equations in the model. In external links exogenous variables in the first submodel is explained by an endogenous variable in the second submodel. In the model presented in this paper, export abroad is the exogenous variable in the quantity (first) model, whereas the price on export abroad is an endogenous variable in the price (second) model. Another example, which has not been included in the model, is the foreign import share, which in the quantity (first) model depends upon foreign import prices compared with domestic prices, which is determined in the price (second) model.

4. The solution to the model

4.1. The model in changes

When we set up a model like the one in this paper, a choice between modelling in absolute variables or in changes arises. The advantage of modelling in absolute variables is that an exact solution of the model can be arrived at. The disadvantage is that it is often necessary to assume linear functions in order to solve the model. The advantage of modelling in changes is that non-linearities can be allowed. Modelling in changes makes all equations linear in these changes. In addition, when modelling in changes the solutions to the model have an interpretation as multipliers. The disadvantage of modelling in changes is that an exact solution of the model is not possible because the model is non-linear. In this paper we choose to model in changes and, thereby, we extend the work by Madsen (2007a & 2007b) to allow for non-linear functions.

In the appendix, the quantity and the price models have been formulated in changes. This is done by differentiating all single equations in both submodels. The solution to the model formulated in changes can now be derived explicitly.

4.2. The solution to the model

The quantity and the price models are solved separately, both by insertion.

In the quantity model the model is solved for unemployment. This involves solutions for employment and for the labour force. The *solution for the labour force* is straightforward involving insertion of equation A.9 into equation A.12, A.10 into A.11 and into A.12. The *solution for the employment* involves a *solution for gross output* and, on the basis of this, *a solution for employment*.

In the *solution for gross output* the intermediate consumption chain is inserted from equation A.1 into equation A.2 and further into A.3 and then into equation A.20. The income, price and *private consumption* chain is then inserted, which involves insertion of equations A.4 to A.9 and further equations A.13 to A.16 and after that into equation A.20. Then, *governmental consumption* in equation A.17 is inserted into A.18, A.19 and then into A.20. Finally, equation A.20 is inserted into A.21, into A.22, into A.23 and finally into A.24.

In *the solution for employment*, which is a part of the solution to the private consumption, two steps of insertion are involved: First, equation A.6 is inserted into A.7, A.4 into A.5 and A.5 into A.7. Then the solution to this subsystem resulting in equation A.7 is inserted into equation A.8 and further into equation A.9 etc.

This process of insertions gives the following result:

$$
dul_g^R = lq_g^R \circ dpop_g^R - Ji'\left[i' - G \circ x \circ ld\,(q_{j,g}^P)^{d-1}\right]^{-1} G \circ g \circ
\left[
\begin{bmatrix}
i - DT(i - d_i^S) \circ S_{IC} B_{IC} b_{IC} \\
- DT(i - d_i^S) \circ S_{CP} B_{CP} b_{CP} \circ (pu_{CP,g})^{-1} \circ pv_g^R \circ v_g^R \circ Ji'\left[i' - G \circ x \circ ld\,(q_{j,g}^P)^{d-1}\right]^{-1} G \circ g
\end{bmatrix}^{-1}
\begin{bmatrix}
+DT(i - d_i^S) \circ S_{CO} B_{CO} b_{CO}\, dpop_g^R \\
+DT(i - d_i^S) \circ du_{I,i}^S \\
+DT(i - dd_i^S) \circ u_i^S \\
\\
+Ddz_i^{P,F} \\
+DT(i - d_i^S) \circ S_{CP} B_{CP} b_{CP} \circ (pu_{CP,g})^{-1} \circ pv_g^R \circ v_g^R \circ Ji'\left[i' - G \circ x \circ ld\,(q_{j,g}^P)^{d-1}\right]^{-1}
\begin{bmatrix}
q_j^P \circ ab\,(pv_{j,g}^P)^{b-1} dpv_{j,g}^P \\
+q_j^P \circ da\,(pv_{j,g}^P)^{b} \\
+G \circ x \circ dl\,(q_{j,g}^P)^{d}
\end{bmatrix} \\
+DT(i - d_i^S) \circ S_{CP} B_{CP} b_{CP} \circ (pu_{CP,g})^{-1} \circ q_g^R \circ py_g^R \circ dv_g^R \\
+DT(i - d_i^S) \circ S_{CP} B_{CP} b_{CP} \circ (pu_{CP,g})^{-1} \circ q_g^R \circ dpv_g^R \circ v_g^R \\
- DT(i - d_i^S) \circ S_{CP} B_{CP} b_{CP} \circ h_g^R \circ (pu_{CP,g})^{-2} dpu_{CP,g} \\
\\
+DT(i - d_i^S) \circ S_{IC} B_{IC} db_{IC} \circ x_j \\
+DT(i - d_i^S) \circ S_{IC} dB_{IC} u_{IC,j}^P \\
+DT(i - d_i^S) \circ dS_{IC} u_{IC,i}^P \\
+DT(i - d_i^S) \circ S_{CP} B_{CP} b_{CP} \circ (pu_{CP,g})^{-1} \circ pv_g^R \circ v_g^R \circ dJq_g^P \\
+DT(i - d_i^S) \circ S_{CP} B_{CP} db_{CP} \circ h_g^R \circ (pu_{CP,g})^{-1} \\
+DT(i - d_i^S) \circ S_{CP} dB_{CP} u_{CP,g} \\
+DT(i - d_i^S) \circ dS_{CP} u_{CP,i}^R \\
+DT(i - d_i^S) \circ S_{CO} B_{CO} db_{CO}\, pop_g^R \\
+DT(i - d_i^S) \circ S_{CO} dB_{CO} u_{CO,g}^R \\
+DT(i - d_i^S) \circ dS_{CO} u_{CO,i}^R \\
+DdTz_i^{S,D} \\
+dDx_i^P
\end{bmatrix}
\right]
$$

$$
+ de\,(pv_g^R)^n \circ pop_g^R
$$

$$
+ en\,(pv_g^R)^{n-1} dpv_g^R \circ pop_g^R \qquad (40)
$$

The price model is solved by straightforward insertion:

Equation A.25 is inserted into equation A.26, then into equation A.27, and finally into equation A.28. Then the insertion line is split into two sublines: The intermediate consumption line involves insertion into equation A.29 and A.30, and finally into equation A.39. The private consumption price-income line involved insertion of equation A.31 further into equations A.32, A.33, A.34 etc. ending inserting into equation A.39. This gives the following solution for the price model:

$$
dpx_j{}' = \begin{bmatrix} i' - DT \circ (i - d_i^{S,F}{}')S_{IC}B_{IC} \circ b_{IC}{}' \\ -DT \circ (i - d_i^{S,F}{}')S_{CP}B_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \end{bmatrix}^{-1}
$$

$$
\begin{bmatrix}
dpz_i^{S,F}{}' \circ d_i^{S,F}{}' S_{IC}B_{IC} \circ b_{IC}{}' \\
+dpz_i^{S,F}{}' \circ d_i^{S,F}{}' S_{CP}B_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
\\
-dul_g{}' c_g^{R}{}' \circ (i' - b_{CP}{}')' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
\\
+px_j{}' dDT \circ (i - d_i^{S,F}{}')S_{IC}B_{IC} \circ b_{IC}{}' \\
+pz_i^{P,D}{}' dT \circ (i - d_i^{S,F}{}')S_{IC}B_{IC} \circ b_{IC}{}' \\
+pz_i^{S,D}{}' \circ (i - dd_i^{S,F}{}')S_{IC}B_{IC} \circ b_{IC}{}' \\
+pz_i^{S,F}{}' \circ dd_i^{S,F}{}' S_{IC}B_{IC} \circ b_{IC}{}' \\
+pu_i^{S}{}' dS_{IC}B_{IC} \circ b_{IC}{}' \\
+pu_{IC,i}^{P}{}' dB_{IC} \circ b_{IC}{}' \\
\\
+px_j{}' dDT \circ (i - d_i^{S,F}{}')S_{CP}B_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pz_i^{P,D}{}' dT \circ (i - d_i^{S,F}{}')S_{CP}B_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pz_i^{S,D}{}' \circ (i - dd_i^{S,F}{}')S_{CP}B_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pz_i^{S,F}{}' \circ dd_i^{S,F}{}' S_{CP}B_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pu_i^{S}{}' dS_{CP}B_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pu_{CP,i}^{R}{}' dB_{BC} \circ b_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
-ul_g{}' d\,c_g^{R}{}' \circ (i' - b_{CP}{}')' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pu_{CP,g}^{R}{}' \circ db_{CP}{}' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pvx_g^{R}{}' \circ (i' - db_{CP}{}')' J_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pv_g^{R}{}' dJ_g^{P,R} \circ i' G_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+pv_g^{P}{}' \circ dG_{j,g}^{P} \circ (i' - jx_j^P{}') \circ (i' - b_{IC}{}') \\
+dpvx_j^{P}{}' \circ jx_j^P{}' \circ (i' - b_{IC}{}') \\
+pve_j^{P}{}' \circ (i' - djx_j^P{}') \circ (i' - b_{IC}{}') \\
+pvx_j^{P}{}' \circ djx_j^P{}' \circ (i' - b_{IC}{}') \\
+pu_{IC,j}^{P}{}' \circ db_{IC}{}' + pv_j^{P}{}' \circ (i' - db_{IC}{}')
\end{bmatrix} \text{............(41)}
$$

5. Interpretation of the solution

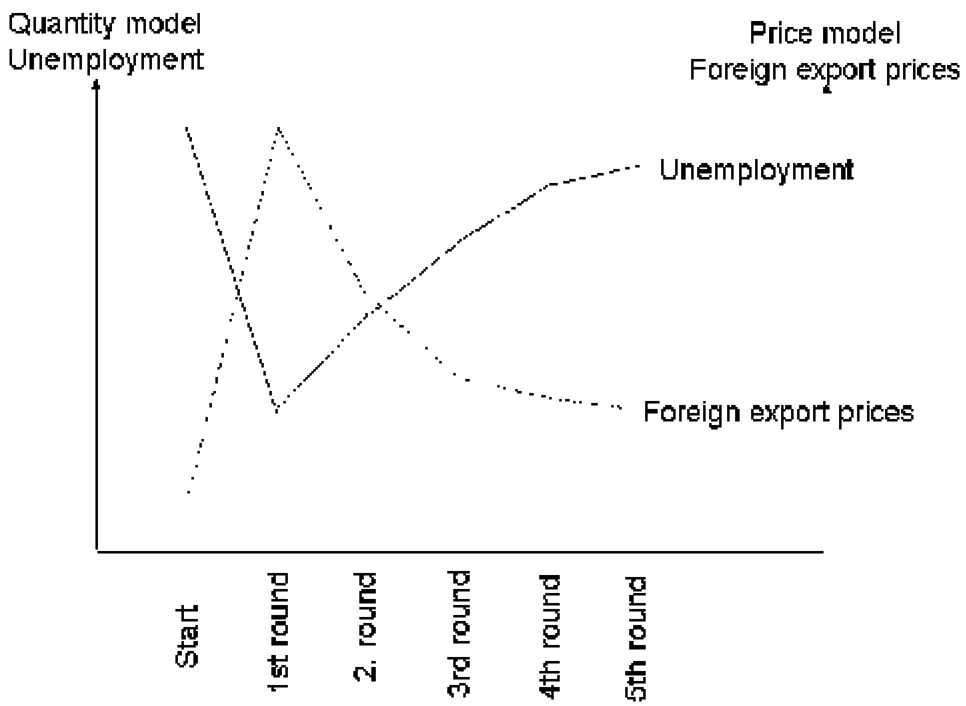
The model and its solutions can now be applied for the analysis of impacts of demographic changes in the local economy. In section 5.1 the impacts are traced through the sequence of calculations starting in the quantity submodel, including both the direct and derived effects (solution equation 40). This is followed by examination of the solution to the price model, involving the direct and derived effects in the price model (solution equation 41). Then a second round solution is obtained involving revised direct and derived effects in the quantity submodel (solution equation 40 recalculated), which is followed by the direct and derived in the price submodel (solution equation 41 recalculated). After a number of calculation rounds involving both submodels the model will converge in a simultaneous solution.

Section 5.2 examines how the impacts on different types of regions depend upon the structure of economic interaction between regions, where the impacts propagate to other regions depending upon the leakages in the local economy. If the economy is very open, the multiplier will be low because leakages in the local economy reduce the impacts, leakages being the trade, the commuting and the shopping, and vice versa for closed local economies multipliers are high.

5.1 Impacts of demographic changes on unemployment and export prices

Starting in the quantity model, the labour supply (l_g^R) decreases as a consequence of the ageing process. This is caused by ageing of population, which leads to retirement. From equation 40 labour force (l_g^R) change, because the number of old people in the population (pop_g^R) increase relatively to the number of young people assuming unchanged age-specific labour participation rates (lq_g^R).

Figure 4. Changes in unemployment and export prices in the iterative steps in the simultaneous solution to the quantity and price models



From equation 40 it can be seen that employment – at least in the short run – will not change due to changes in population, except for minor adjustments in governmental consumption and private consumption. Adjustments in governmental consumption will lead to minor increases in employment, because average governmental consumption for old people can be expected to be higher than for young people, which in turn leads to higher demand, production and employment. On the other hand, private consumption share will decrease as it can be assumed, that the share will decrease due to lower consumption rate with increasing age. The net effect will be small changes in total consumption rate.

The net effect of ageing is therefore that, ceteris paribus, unemployment – in the short run – will decrease because of the ageing process. Figure 4 illustrates the changes in unemployment in the 1st round calculation in the quantity model (the minimum point on the non-dotted curve).

Although, the ageing process does not involve employment reduction, the structural shift from productive sectors to increasing production of services for the elder population should be added. This specialisation increases the vulnerability of age-dependent regions and local economies, because the export base is downscaled.

Now, moving to the price model, decreases in unemployment (dul_g^R) will lead to higher nominal income rates and in turn to higher prices on production (see equation 41). This will – at least in the short run – leave the real domestic income rate unchanged. But the net effect of ageing will be higher producer prices, which in turn have impacts on prices for exports abroad and domestic prices compared with prices for import from abroad. The initial increase in the foreign export prices is illustrated in Figure 4 (the maximum point on the dotted curve). In the quantity model reduced demand from lower foreign exports and higher imports from abroad modify the reduction in unemployment, because employment decreases. This increase in 2nd round calculation of the unemployment is illustrated in figure 4. This in turn will lead to a decrease in export prices, which is illustrated in figure 4 as the result of the 2nd round calculation.

After a number of calculation rounds the unemployment and prices converge to a new equilibrium, where unemployment is back at the original level and relative export prices are at a higher level than the initial level of prices. As a consequence, exports decrease and import increases and employment will go down.

This process will lead to changes in both real income rate and employment, together having a negative net effect on total income. Real income rate can be assumed to be unchanged because both the nominal income rates and the prices are expected to increase, leaving the real income rates unchanged. Employment will decrease, because the labour force will decrease markedly. With a reduction in employment the unemployment will be unchanged. Net real income will go down because employment times real income rates will decrease.

5.2. Regional impacts of ageing

Using the simultaneous solution to the two submodels, it can be seen that the regional impacts of a given ageing process depend upon the leakages in the local economy. The more open the economy, the less the negative impacts will be on the local area itself, exporting the negative impacts to other regions and the rest of the world. For closed economies, the negative impacts will be multiplied, because most impacts are in the local area itself. Taking trade as an example, the negative impacts of a reduction in foreign exports and increase in import from abroad on the local area, where the ageing takes place, will be smaller the higher the share of interregional and international imports.

This truncation in the negative impacts from ageing due to leakages in the local economy applies both in the quantity model and the price model. In the quantity model, the reduction in production due to reductions in demand will be smaller, the higher the import share, the higher out-commuting and the higher the shopping outside the local area. In the price model, similarly the increases in prices due to income increases in the local area will be smaller the more open economy. The higher the share of export abroad and exports to other regions, the smaller the spill-over of cost increases from rising income will be. The higher the sale of commodities to consumers from outside the local area, the more consumers from outside will bear the burden from increasing consumer prices. And the higher the share of the employed, who resides outside the local area, the more the negative impacts from rising income cost on production will be exported to other regions.

This analysis of the multipliers in the local economies does not apply to trade alone, but to all types of interactions in the model, such as shopping and commuting. Taking the multipliers in equations 40 and 41 as the point of departure the leakages, and the impacts on different types of local economies, are the following:

From Table 1, 7 types of local areas have been defined: Metropolitan areas divided into the centre, residential suburbs and production suburbs, urban centres, divided into urban and suburban areas, and finally rural areas, divided into areas which are connected or are not connected to urban centres.

For shopping in intermediate consumption goods (column 1) commodities are to some degree bought outside the area (indicated by “Mid” in row 1). For the centre of metropolitan areas, this share can be assumed to be lower because wholesaling activities can be assumed to be concentrated in metropolitan areas. Therefore the intra-shopping is relatively high, marked with a “++” in row 2. For suburban areas, which are specialised in production activities probably a number of wholesale firms might be located, although the probability is much lower than for metropolitan centre areas. Therefore, a ”+” has been indicated in row 3. For residential metropolitan areas no wholesale activities can be expected, and therefore an “--“ is shown in row 4. For urban centres, wholesale functions are found, but not as frequently as in metropolitan areas, giving a “+” in row 5. For rural areas wholesale activities or firms supplying intermediate consumptions goods for rural firms are underrepresented leading to a “--“ in rows 6 and 7.

In column 2 the leakages originating from commuting are shown. In general, leakages working through income formation are substantial. Leakages are smaller in the centre of the metropolitan area, because relative fewer have jobs outside the area, whereas residential areas have relative more jobs outside the area. Especially, for rural areas disconnected from urban centres commuting out of the area is very limited.

In column 3 the leakages from shopping is shown and is in general being at medium level. Leakages are smaller for urban metropolitan areas, higher for residential areas and low for rural areas disconnected from urban centres.

In column 4 leakages for shopping in governmental consumption are examined, being at a low level, because almost all individual governmental consumption is produced and consumed at the local level. General governmental consumption and specific types of individual governmental consumption such as hospitals and educational services are supplied outside the local area. From a regional point of view, because general governmental consumption and the specific individual governmental consumption are geographically concentrated in metropolitan areas, leakages are lowest in the centre of the metropolitan area and decreasing the longer away from the city centres.

Table 1. Geographical leakages and typology for local economies

Type of region		Shopping in intermediate consumption goods in same region	Commuting to same region	Shopping in private consumption in same region	Shopping in governmental consumption in the same region	Domestic supply / local demand	Local supply / domestic sales	Total multiplier
Mathematical representation		$\frac{u_{IC,i}^{P,S} \parallel P=S}{u_{IC,i}^{P}}$	$\frac{q_{g}^{P,R} \parallel P=R}{q_{g}^{P}}$	$\frac{u_{CP,i}^{R,S} \parallel R=S}{u_{CP,i}^{R}}$	$\frac{u_{CO,i}^{R,S} \parallel R=S}{u_{CO,i}^{R}}$	$\frac{z_{i}^{D} \parallel S=P}{u_{i}^{S}}$	$\frac{z_{i}^{S,P} \parallel S=P}{u_{i}^{S,D}}$	
National average		Mid	High	Mid	Low	Mid	Low	
Metropolitan area:	Centre	++	++	++	+	0	--	High
	Residential suburbs	--	--	--	-	0	--	Very low
	Production suburbs	+	++	-	-	0	++	High
Urban centres	Centre	+	+	+	+	0	+	High
	Suburbs	-	--	-	-	0	-	Very low
Rural areas	Connected to urban centres	--	--	--	-	0	--	Very low
	Disconnected from urban centres	--	++	++	0	0	--	Mid

In column 5 and 6 the trade leakages are examined. In general, these leakages are much higher than leakages for shopping and commuting. Relatively, leakages are lowest in productive areas such as urban centres and production suburbs and highest in metropolitan centres and in rural areas.

Concluding, the total multiplier (see column 7), both in terms of the quantity model and the price model is highest in productive suburban areas, where the leakages are smallest, especially in trade, whereas multipliers are lowest in residential areas, where leakages are highest both in trade and in commuting and shopping. The metropolitan centre has below average leakages, being very low in commuting and shopping leakages and at medium level in trade. The multiplier in rural areas, with connection to urban centres will be very low, whereas leakages in rural areas will be medium in rural areas where leakages are small in commuting and shopping, but high in trade.

5.3. Specification of the model

The basic model, which has been presented above, represents specific equilibrium and behavioural mechanism, reflecting a short time horizon and equilibrium in the commodity market, but not in the labour market. Even though these seemingly restrictive characteristics the model can be adjusted in order to reflect other time horizons and behavioural mechanism. In relation to the demographic problem the following characteristics of the basic model – and relevant extensions – seem important:

Three types of regions with different functions in the economy are represented in the model: The region itself, other regions and regions abroad. In the case of Denmark, it could be Danish municipalities, Denmark and countries outside Denmark. In the case of the European Union, it could be the regions, the EU and the Rest of the World. The regional structure, which the model should reflect, is of course important for decisions concerning model structure. The division into three types of region seems relevant and important from a policy and a behaviour point of view.

In the basic model there is not full treatment of interaction: Only trade involves both interregional and international trade, whereas commuting and shopping only include interregional interaction, whereas international commuting and shopping have been excluded from the model. As internationalization evolves, it is relevant to extend the model to include international commuting and shopping/tourism. In the formulation of the general interregional quantity model (Madsen 2007a) and the general interregional price model (Madsen 2007b) all types of interaction, international as well as interregional have been included into the model.

In the basic model, regional wealth, interest payments and balance of payment – both the international and the interregional – do not enter. The regional wealth, interest and balance of payment issues involve the time horizon, where an extension of the model with wealth and interest payments will include the medium or even long run by taking into account the adjustments in the regional economy to the reduction in wealth, derived interest payments, which regions with ageing problems face. Another alternative would be to assume flexible exchange rates, but this only undertakes the international balance of a payment adjustment process.

The basic model implicitly assumes that the demographic development only originates from the region itself. Interregional as well as international migration should be added, if medium or long-term adjustments are to be included. Migration will change the population and the labour force, which will have an impact on unemployment and income rates, which in turn will have an impact on employment and participation rates. Although modelling of migration behaviour is complex, inclusion of impacts of migration is straightforward. The change in population can simply be interpreted as including migration plus population changes from the region itself.

6. Conclusion

In this paper, the impacts of the population ageing process on the regional and local economy are discussed in the framework of the general interregional quantity and price models (Madsen 2007a & 2007b). The ageing process identifies decreasing participation rates for the labour force and decreasing productivity in production. The negative impacts depend upon the direct changes in population by age group, the direct labour market effects of population changes and the derived or multiplier effects from these changes.

In the quantity submodel the composition of labour force is determined by relative income rates. Participation rates, productivity and the composition of employment are assumed to be iso-elastic. To evaluate the impacts of ageing of such iso-elastic behaviour the general interregional quantity and price submodels have been formulated in changes. In this way, the model can be used to evaluate effects of changes in the locale economies from the ageing of population assuming more consistent behaviour than in the pure linear models. The quantity and price submodels have been solved separately, and the simultaneous solution is discussed. The multipliers of the model are evaluated theoretically, using plausible assumption about leakages in the local economy. Based upon these theoretical and empirical arguments multipliers for the centre and for the production suburbs in metropolitan areas and for rural areas disconnected from urban areas seems to have multipliers above average, whereas multipliers for the residential areas, both in metropolitan and urban areas have below average multipliers. Multipliers involve both the conventional quantity model multipliers as well as the multipliers derived from the price model.

Appendix

The general interregional quantity model in growth form

$du_{IC}^{P} = b_{IC} \circ dx_j + db_{IC} \circ x_j$ ....(A.1)

$du_{IC,i}^{P} = B_{IC} du_{IC}^{P} + dB_{IC} u_{IC}^{P}$ ....(A.2) *from j to i*

$du_{IC,i}^{S} = S_{IC} du_{IC,i}^{P} + dS_{IC} u_{IC,i}^{P}$ ....(A.3) *from P to S*

$dg = d\, l\, (q_{j,g}^{P})^{d} + l\, d\, (q_{j,g}^{P})^{d-1} dq_{j,g}^{P}$ ....(A.4)

$dq_{j}^{P} = x_j \circ dg + dx_j \circ g$ ....(A.5) *from j to g*

$dG = da\, (pv_{j,g}^{P})^{b} + a\, b\, (pv_{j,g}^{P})^{b-1} dpv_{j,g}^{P}$ ....(A.6)

$dq_{j,g}^{P} = G \circ dq_{j}^{P} + dG \circ q_{j}^{P}$ ....(A.7) *from j to g*

$dq_{g}^{P} = i' dq_{j,g}^{P}$ ....(A.8) *from P to R*

$dq_{g}^{R} = J dq_{g}^{P} + dJ q_{g}^{P}$ ....(A.9) *from P to R*

$dlq_{g}^{R} = en\, (pv_{g}^{R})^{n-1} dpv_{g}^{R} + de\, (pv_{g}^{R})^{n}$ ....(A.10)

$dl_{g}^{R} = lq_{g}^{R} \circ dpop_{g}^{R} + dlq_{g}^{R} \circ pop_{g}^{R}$ ....(A.11)

$dul_{g}^{R} = dl_{g}^{R} - dq_{g}^{R}$ ....(A.12)

$dh_{g}^{R} = dq_{g}^{R} \circ pv_{g}^{R} \circ v_{g}^{R} + q_{g}^{R} \circ dpv_{g}^{R} \circ v_{g}^{R}$
$+ q_{g}^{R} \circ pv_{g}^{R} \circ dv_{g}^{R}$ ....(A.13)

$du_{CP,g}^{R} = b_{CP} \circ dh_{g}^{R} \circ (pu_{CP,g})^{-1} + db_{CP} \circ h_{g}^{R} \circ (pu_{CP,g})^{-1}$
$- b_{CP} \circ h_{g}^{R} \circ (pu_{CP,g})^{-2} dpu_{CP,g}$ ....(A.14)

$du_{CP,i}^{R} = B_{CP} du_{CP,g} + dB_{CP} u_{CP,g}$ ....(A.15) *from g to i*

$du_{CP,i}^{S} = S_{CP} du_{CP,i}^{R} + dS_{CP} u_{CP,i}^{R}$ ....(A.16) *from R to S*

$du_{CO,g}^{R} = b_{CO} dpop_{g}^{R} + db_{CO}\, pop_{g}^{R}$ ....(A.17)

$du_{CO,i}^{R} = B_{CO} du_{CO,g}^{R} + dB_{CO} u_{CO,g}^{R}$ ....(A.18) *from g to i*

$du_{CO,i}^{S} = S_{CO} du_{CO,i}^{R} + dS_{CO} u_{CO,i}^{R}$ ....(A.19) *from R to S*

$du_{i}^{S} = du_{IC,i}^{S} + du_{CP,i}^{S} + du_{CO,i}^{S} + du_{I,i}^{S}$ ....(A.20)

$dz_{i}^{S,D} = (i - d_{i}^{S}) \circ du_{i}^{S} + (i - dd_{i}^{S}) \circ u_{i}^{S}$ ....(A.21)

$dz_{i}^{P,D} = T dz_{i}^{S,D} + dT z_{i}^{S,D}$ ....(A.22) *from S to P*

$dx_{i}^{P} = dz_{i}^{P,D} + dz_{i}^{P,F}$ ....(A.23)

$dx_j = D dx_{i}^{P} + dD x_{i}^{P}$ ....(A.24) *from i to j*

$$dpx_i^P{}' = dpx_j{}'D + px_j{}'dD \text{......} (A.25) \quad \textit{from } j \textit{ to } i$$

$$dpz_i^{P,D}{}' = dpx_i^P{}' \text{......} (A.26)$$

$$dpz_i^{S,D}{}' = dpz_i^{P,D}{}'T + pz_i^{P,D}{}'dT \text{......} (A.27) \quad \textit{from } P \textit{ to } S$$

$$dpu_i^S{}' = dpz_i^{S,D}{}' \circ (i - d_i^{S,F}{}') + dpz_i^{S,F}{}' \circ d_i^{S,F}{}'$$

$$+ pz_i^{S,D}{}' \circ (i - dd_i^{S,F}{}') + pz_i^{S,F}{}' \circ dd_i^{S,F}{}' \text{......} (A.28)$$

$$dpu_{IC,i}^P{}' = dpu_i^S{}'S_{IC} + pu_i^S{}'dS_{IC} \text{......} (A.29) \quad \textit{from } S \textit{ to } P$$

$$dpu_{IC,j}^P{}' = dpu_{IC,i}^P{}'B_{IC} + pu_{IC,i}^P{}'dB_{IC} \text{......} (A.30) \quad \textit{from } i \textit{ to } j$$

$$dpu_{CP,i}^R{}' = dpu_i^S{}'S_{CP} + pu_i^S{}'dS_{CP} \text{......} (A.31) \quad \textit{from } S \textit{ to } R$$

$$dpu_{CP,g}^R{}' = dpu_{CP,i}^R{}'B_{CP} + pu_{CP,i}^R{}'dB_{BC} \text{......} (A.32) \quad \textit{from } i \textit{ t o } g$$

$$dpvx_g^R{}' = (dul_g^n{}' - dul_g{}')c_g^R{}' + (ul_g^n{}' - ul_g{}')dc_g^R{}' \text{......} (A.33)$$

$$dpv_g^R{}' = dpu_{CP,g}^R{}' \circ b_{CP}{}' + dpvx_g^R{}' \circ (i' - b_{CP}{}')'$$

$$+ pu_{CP,g}^R{}' \circ db_{CP}{}' + pvx_g^R{}' \circ (i' - db_{CP}{}')' \text{......} (A.34)$$

$$dpv_g^P{}' = dpv_g^R{}'J_g^{P,R} + pv_g^R{}'dJ_g^{P,R} \text{......} (A.35) \quad \textit{From } Rg \textit{ to } Pg$$

$$dpve_{g,j}^P{}' = dpv_g^P{}' \circ G_{j,g}^P + pv_g^P{}' \circ dG_{j,g}^P \text{......} (A.36) \quad \textit{From } Pg \textit{ to } Pj$$

$$dpve_j^P{}' = i'dpve_{g,j}^P \text{......} (A.37)$$

$$dpv_j^P{}' = dpve_j^P{}' \circ (i' - jx_j^P{}') + dpvx_j^P{}' \circ jx_j^P{}'$$

$$+ pve_j^P{}' \circ (i' - djx_j^P{}') + pvx_j^P{}' \circ djx_j^P{}' \text{......} (A.38)$$

$$dpx_j{}' = dpu_{IC,j}^P{}' \circ b_{IC}{}' + dpv_j^P{}' \circ (i' - b_{IC}{}')$$

$$+ pu_{IC,j}^P{}' \circ db_{IC}{}' + pv_j^P{}' \circ (i' - db_{IC}{}') \text{......} (A.39)$$

Literature

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